THIRD INTERNATIONAL CONFERENCE ON SPACE STRUCTURES

Edited by

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ELSEVIER APPLIED SCIENCE PUBLISHERS
LONDON and NEW YORK

Proceedings of the Third International Conference on Space Structures, held at the University of Surrey, Guildford, UK, 11–14 September 1984.

STRUCTURING: A PROCESS OF MATERIAL DILUTION

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The concept that structuring is a process of material dilution is introduced by first considering a simple structural cell and showing that its stiffness can be represented by a piece of solid material. Comparing the representative material properties with the actual material properties reveals a process of dilution. The process is shown to be recursive. Structural performance is related to representative material properties, and the Strut Problem is used to illustrate how the dilution concept can be utilised in design.

INTRODUCTION

Advances in structural analysis over recent years have been associated with the formulation of matrix methods, Ref 1, the development of finite element theory, Ref 2, and the use of computers to execute the programs embodying these techniques. Elastic stresses anywhere in any structure can now be calculated as accurately as required using commercially available computer software. In the early 1940s, Hrennikoff, Ref 3 and McHenry, Ref 4, were not in this position, but they were able to analyse frameworks and perceived that there was an analogy between solid material and a regular lattice having the same envelope, and they utilised it to analyse 2 and 3d stress problems.

Modern finite elements can be viewed as a development of Hrennikoff's framework method. No analogy is evident, as stiffness matrices derived from continuum theory of elasticity have long been substituted for frameworks, but structural analogies are still being found useful. Bridge decks are often modelled as beams or grillages of beams, Ref 5, and space frames as plates, Ref. 6. When complex structures like these can be modelled in a simpler way, their analysis is simpler.

This paper considers the converse of Hrennikoff's framework analogy, namely that every regular structure is analogous to a solid piece of material having nearly the same envelope. Analysis of the solid model is simpler though possibly not so accurate, and the derived material properties of the "material" of the solid analogue may be considered as the properties of the structure. Using photographic terminology Hrennikoff was zooming in to his structure, looking for accuracy in analysis. The converse is to zoom out to take a distant view, to consider the continuum properties of the structure as a whole. These are relevant at the design stage when considering structural form.

Comparison of these structural properties with the continuum properties of the material of which the structure is made suggests that structuring is a process of material dilution. The second section introduces this concept with a development of simple beam theory, focusing on the performance of structural members. Then structuring is looked at from a material viewpoint in the third section. Finally, both views are united in considering whole structures and their performance.

SIMPLE BEAM THEORY

A repetitive structure lies along the line AB of Fig 1a parallel to the x axis. A single cell of length L lies between Sections 1 and 2. This cell is supported at Section 1 and loaded at Section 2 by an axial force P_2, a shear Q_2 and a moment M_2. When the load distribution within the whole cell is known, the axial extension u_2, the transverse displacement v_2 and the rotation of the section θ_2 are derivable by Castigliano's theorem as:

$$u_2 = \partial U/\partial P_2$$
$$v_2 = \partial U/\partial Q_2$$
$$\theta_2 = \partial U/\partial M_2 \qquad \ldots\ldots 1.$$

U, the internal energy, is a function of P_2, Q_2 and M_2 and the geometry of the structural cell. These three equations may be written as:

$$\begin{bmatrix} u_2 \\ v_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} P_2 \\ Q_2 \\ M_2 \end{bmatrix} \qquad \ldots\ldots 2.$$

or simply as $d_2 = FR_2$ ..... 3.

where d_2 is the displacement vector, R_2 is the load vector and F is a flexibility matrix.

By the reciprocal theorem F is symmetric, i.e. $f_{21} = f_{12}$ etc. Therefore the unit cell has six flexibilities: f_{11}, f_{22}, f_{33}, f_{12}, f_{13} and f_{23}.

If the cell were symmetrical about AB, reflection of the cell about this axis, as shown in Fig 1b, would not affect the flexibility at end 2. Related to the cell's local axis system:

$$\begin{bmatrix} u_2 \\ -v_2 \\ -\theta_2 \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{12} & f_{22} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{bmatrix} \begin{bmatrix} P_2 \\ -Q_2 \\ -M_2 \end{bmatrix} \qquad \ldots\ldots 4.$$

from which it follows that:

$$f_{12} = f_{13} = 0 \qquad \ldots\ldots 5.$$

If the cell were symmetrical about a perpendicular axis CD, reflection of the cell about this axis, as shown in Fig 1c, would not affect the flexibility at end 2 either. Then:

$$\begin{bmatrix} u_2 \\ L\theta_2 - v_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{12} & f_{22} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{bmatrix} \begin{bmatrix} P_2 \\ -Q_2 \\ M_2 + Q_2 L \end{bmatrix}$$

from which it would follow that:

$$f_{12} = \tfrac{1}{2}Lf_{13} \quad \ldots\ldots 6.$$

and $$f_{23} = \tfrac{1}{2}Lf_{33} \quad \ldots\ldots 7.$$

Hence, with both these axes of symmetry, there are only three flexibilities, f_{11}, f_{22} and f_{33} with the other three defined by Eqns 5 and 7. Cells with these properties will be referred to as orthotropic.

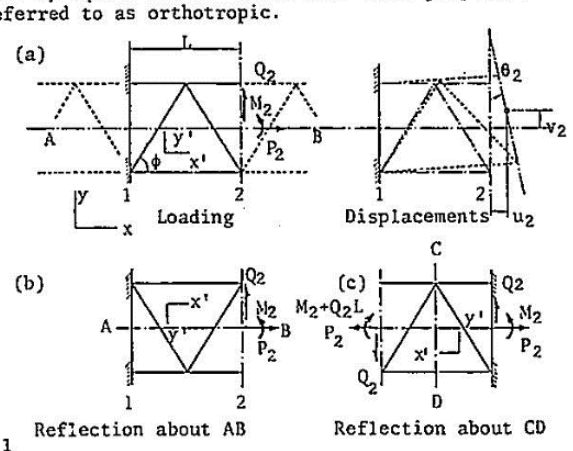


Fig 1

The frame shown in the unit cell of Fig 1 is not symmetric about AB, yet does behave orthotropically when the framework is pin-jointed. Because it is statically determinate, member loading is simple to calculate and the displacement vector relates to the points where the loading is applied. For such cases Eqn 3 is exact for small displacements. For redundant cell structures assumptions need to be made about the internal distribution of load in order to be able to calculate F. Related to this difficulty is an uncertainty about the meaning of the displacement vector; for example, an initially plane section will not generally remain plane when loaded.

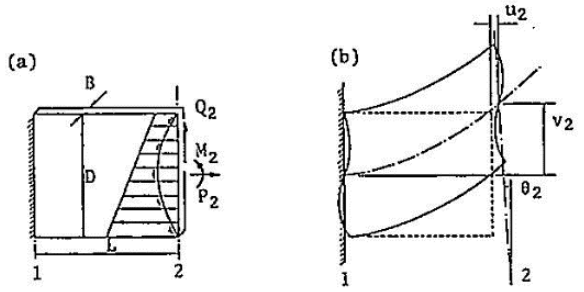


Fig 2

Consider a solid rectangular section as an example of a redundant cell structure. As shown in Fig 2a, the cell has dimensions L, B and D. The loading at end 2 is assumed to be distributed through the cell according to simple beam theory as illustrated. Using this distribution to determine U, the displacement vector at end 2 may be calculated from Eqn 1. The displacement vector describes a displaced position for section 2, but cannot describe how the section is curved as illustrated in Fig 2b, due to the parabolic variation of shear stress. It is consistent with simple beam theory to ignore this incompatibility and to assume sections always remain plane, even under shear.

The three flexibility coefficients f_{11}, f_{22} and f_{33} can be calculated from Eqn 1 as:

$$f_{11} = L/EA$$
$$f_{22} = 1.2L/GA + L^3/3EI$$
$$f_{33} = L/EI \qquad \text{where:} \quad \ldots\ldots 8.$$

E = Young's modulus of the material
G = Shear modulus of the material
A = Sectional area, BD
I = "Inertia" of the section, $BD^3/12$

The flexibility matrix of any orthotropic cell of length L may be written in the following form:

$$\begin{bmatrix} L/k_1 & 0 & 0 \\ 0 & L/k_2+L^3/3k_3 & L^2/2k_3 \\ 0 & L^2/2k_3 & L/k_3 \end{bmatrix} \quad \ldots\ldots 9.$$

where k_1, k_2 and k_3 are principal stiffnesses. For the solid rectangular section:

$$k_1 = EA$$
$$k_2 = GA/1.2$$
$$k_3 = EI \quad \ldots\ldots 10.$$

From the flexibility matrix F, which relates d_2 to R_2 by Eqn 3, a transfer matrix T_L can be derived which relates d_2 and R_2 to d_1 and R_1. d_2 and R_2 are concatenated to form a state vector S_2 of six elements which fully defines the displacements and internal reactions at section 2. S_1 is defined similarly. The sign convention used is consistent with Fig 1, showing displacements and forces in a positive sense. The forces are those which the greater x side applies to the lesser x side. When the cell is orthotropic, the state vectors are related as follows:

$$\begin{bmatrix} u_2 \\ v_2 \\ \theta_2 \\ P_2 \\ Q_2 \\ M_2 \end{bmatrix} = \left[\begin{array}{c|c|c|c|c|c} 1 & 0 & 0 & L/k_1 & 0 & 0 \\ 0 & 1 & L & 0 & L/k_2-L^3/6k_3 & \tfrac{1}{2}L^2/k_3 \\ 0 & 0 & 1 & 0 & -\tfrac{1}{2}L^2/k_3 & L/k_3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -L & 1 \end{array}\right] \begin{bmatrix} u_1 \\ v_1 \\ \theta_1 \\ P_1 \\ Q_1 \\ M_1 \end{bmatrix} \quad \ldots\ldots 11.$$

or: $S_2 = T_L S$ $\quad \ldots\ldots 12.$

By multiplying the two matrices term by term, it can be shown that, taking L as the only variable in T_L:

$$T_X T_Y = T_{X+Y} \quad \ldots\ldots 13.$$

for any values of X and Y. Therefore the transfer matrix for n cells strung together is given by:

$$(T_L)^n = T_{nL} \quad \ldots\ldots 14.$$

Hence, Eqn 11 defines the transfer matrix of a member made up of any number of cells when L is redefined as the member's length. It follows that the deflections of such a member at the ends of each cell can be described by simple beam theory, and for orthotropic celled members, the flexibility of the member is defined by the three stiffnesses k_1, k_2 and k_3 which are properties of the constituent material and the cell geometry.

The same set of stiffnesses defines the structural behaviour of a family of different cross-sections and cell structures. It would seem appropriate to represent all these different sections by the simplest, such as a circular, square or rectangular solid section. Denoting the representative properties by a prime, the axial and flexural properties are represented correctly when:

$$E'A' = k_1 = EA$$
$$E'I' = k_3 = EI \quad \ldots\ldots 15.$$

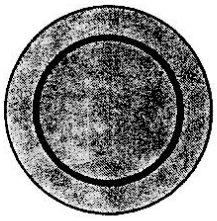
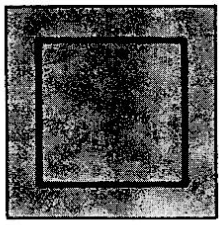
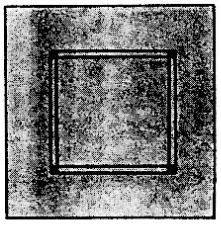
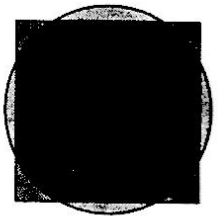
$$\therefore I'/A' = k_3/k_1 = I/A$$
$$\therefore r'^2 = k_3/k_1 = r^2 \quad \ldots\ldots 16.$$

Where r denotes the radius of gyration. Hence, the radius of gyration of the representative solid section can be determined, and after choosing a shape for the section, its size can also be determined. Then A' is known, and from Eqn 15, E' can be calculated.

The area A in the axial stiffness EA usually has physical significance as the net sectional area of material which is axially oriented; in a lattice member for example, the bracing does not contribute significantly to A. It will be convenient to introduce:

$$i = E/E' = A'/A \quad \ldots\ldots 17.$$

While material on its own makes up the area A, both material and space make up A', which is why it is appropriate to call i a dilution factor. Fig 3 shows various sections with their representative solid sections shown shaded. The

Section:	Circular tube dia. d. thk. t	Square tube side d, thk. t	Square lattice side d, txt	Solid square side d
Representative Section:	Solid circle dia. D	Solid square side D	Solid square side D	Solid circle dia. D
D/d	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{3}$	$2/\sqrt{3}$
Dilution factor	$\frac{1}{2}d/t$	$\frac{1}{2}d/t$	$0.75(d/t)^2$	$\pi/3$

Fig 3

principal geometric relationships and dilution factors are written beneath each.

The shear stiffness k_2 can be linked to the representative section by calculating a shear modulus G', or else it can be linked to k_1 or EA in the following way:

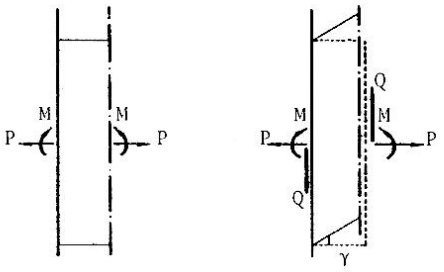


Fig 4

Fig 4 shows a short length δL of representative beam, whose behaviour is governed by Eqn 11. Since δL is small, any terms associated with L^2 or L^3 can be ignored, and therefore moments do not induce significant sway. An axial load P, tensile being positive, is applied to the section, together with a moment M, when a shear Q is subsequently applied which produces a sway rotation of γ. If the internal load distribution produced by the sway is unaffected by the loads P and M, then the increase in internal energy due to the sway is given by:

$$\delta U = \tfrac{1}{2}k_2\gamma^2\delta L \qquad \ldots\ldots 18.$$

Work done by the force Q is given by:

$$\delta E_Q = \tfrac{1}{2}Q\gamma\delta L \qquad \ldots\ldots 19.$$

Work done by the applied loads P and M is given by:

$$\delta E_P = P\delta L(1-\cos\gamma) = -\tfrac{1}{2}P\gamma^2\delta L \qquad \ldots\ldots 20.$$

If the strain produced by P is ε, then:

$$P = k_1\varepsilon \qquad \ldots\ldots 21.$$

Equating δU to $\delta E_Q + \delta E_P$, the apparent sway stiffness k_S is given by:

$$k_S = Q/\gamma = k_2 + k_1\varepsilon \qquad \ldots\ldots 22.$$

Hence, when ε is zero, $k_S = k_2$, but as P increases, so does the apparent sway stiffness. This tension stiffening effect has been shown to account for torsional stiffening of bars under tension, Ref 7. Eqn 22 is of particular relevance to structural members when k_2 may be much smaller than k_1. Then a sway instability is predicted when:

$$\varepsilon = -\varepsilon_S \text{ say} \quad = -k_2/k_1 \qquad \ldots\ldots 23.$$

because k_S vanishes at this axial strain.

It is therefore suggested that the shear stiffness k_2 should be regarded as $k_1\varepsilon_S$. For solid members made of isotropic material, ε_S is always greater than 0.4. At such strains, simple first and second order theories are not applicable, so Eqn 23 does not predict instablity in such materials. Lightly braced lattice members may have quite low values of ε_S, at which strain Eqn 23 does predict instability, however stocky the member might be.

Flexural instability of members is described by Euler's equation as occurring at the Euler load P_e given by:

$$P_e = \pi^2 EI/L^2 \qquad \ldots\ldots 24.$$

ignoring the effects of shear flexibility. For a solid circular section of diameter D, Eqn 24 may be written as:

$$\varepsilon_e = \frac{\pi^2}{16}\frac{D^2}{L^2} \qquad \ldots\ldots 25.$$

revealing that the Euler strain is a function only of slenderness L/D, and is not influenced by the properties of the material within the member.

The effect of shear flexibility on member stability is considered in Ref 8. Manipulation of the formula given in this reference yields a critical buckling strain, ε_c, given by:

$$\frac{1}{\varepsilon_c} = \frac{1}{\varepsilon_e} + \frac{1}{\varepsilon_S} \qquad \ldots\ldots 26.$$

If ε_c is the critical strain a member is designed to, then attention must be given to the critical strains of the component members of that member, to ensure that they are also sufficiently stable at ε_c.

In this section, it has been shown how orthotropic structuring of an orthotropic material into a line member can be represented by an equivalent solid line member made of a derived hypothetical orthotropic material, having more dilute properties than the original material. Compressive performance has been shown to be limited by sway and flexural instabilities.

MATERIAL DILUTION

The structuring of material implies the formation of surfaces separating the space into those parts occupied by material and those parts that are not. When material and space are mixed together in this way, the mixture may be regarded either as voided or as dispersed material; either way, the material has to be shaped, and this process can be described as structuring. The words "nodal", "lineal" and "surface" have been chosen to describe the formation of a void or a piece of material about a point, a line or a surface respectively.

Suppose material is dispersed uniformly in space, lineally, with all lines of material oriented parallel to the x_1-axis. If the boundaries of this space are displaced and distorted by a uniform strain field, defined by (ε_{11}, ε_{22}, ε_{33}, γ_{12}, γ_{13}, γ_{23}) then the stress in the material is given by:

$$\sigma_{11} = E\varepsilon_{11} \quad \ldots\ldots 27.$$

where E is its Young's modulus. The whole volume can be represented by a solid material stressed only in the x_1 direction by a stress:

$$\sigma_{11}' = E'\varepsilon_{11} = mE\varepsilon_{11} \quad \ldots\ldots 28.$$

where:

m is the proportion of the total volume occupied by material. m will be referred to as a material concentration factor. The other five components of stress are zero.

Suppose material is dispersed uniformly in planes parallel to the x_1 and x_2 axes, then the representative properties of the solid filling the whole space are given by:

$$\begin{aligned} E_{11}' &= mE_{11} \\ E_{22}' &= mE_{22} \\ G_{12}' &= mG_{12} \\ \nu_{12}' &= \nu_{12} \end{aligned} \quad \ldots\ldots 29.$$

The density of the representative material ρ' is also related to that of the actual material ρ by:

$$\rho' = m\rho \quad \ldots\ldots 30.$$

and the dilution factor i, as defined in Eqn 17, can have six values:

$$\begin{aligned} i_{11} &= E_{11}/E_{11}' = 1/m \\ i_{22} &= E_{22}/E_{22}' = 1/m \\ i_{12} &= G_{12}/G_{12}' = 1/m \\ i_{33} &= i_{13} = i_{23} = \infty \end{aligned} \quad \ldots\ldots 31.$$

When the space is strained in the x_3 direction, there is no resistance; no material is stressed. This may be considered a structuring inefficiency and η may be considered a structural efficiency factor defined as:

$$\eta = 1/mi \quad \ldots\ldots 32.$$

At best η can be one. When m is one, there is no structuring, just solid material, and all six ηs are one, indicating no inefficiency. Any structuring of material will introduce a degree of inefficiency. In the example above, three ηs equal unity, but three are zero. This example is only efficient for transmitting loading in the x_1x_2 plane.

By combining material dispersed in several different orientations, the structuring becomes more practical, possessing stiffness in all directions. Its representative properties may be simply calculated by transforming the local stiffness matrix of each set of material to a global axis system. Summing them and inverting the resulting stiffness matrix yields the representative flexibility matrix, and hence the required properties. Material property transformations are described in Refs 9 and 10.

Fig 5a shows a random distribution of straight lines on a plane. This configuration usefully illustrates some general principles of structuring. Consider it first as being a section through a set of planes parallel to the x_1 axis. When material is dispersed about these planes, it forms a prismatic structure with axisymmetric properties. Using the method described above, it can be shown that:

$$\begin{aligned} E_{11}' &= mE_{11} \\ E_{22}' &= mE_{22}/(3-2\alpha\nu_{12}^2) \\ G_{12}' &= mG_{12}/2 \\ \nu_{12}' &= \nu_{12} \\ \nu_{23}' &= (1-2\alpha\nu_{12}^2)/(3-2\alpha\nu_{12}^2) \\ G_{23}' &= mE_{22}/8(1-\alpha\nu_{12}^2) \end{aligned} \quad \ldots\ldots 33.$$

where $\alpha = E_{22}/E_{11}$

from which the following material efficiencies may be deduced:

$$\begin{aligned} \eta_{11} &= 1 \\ \eta_{12} &= 1/2 \\ \eta_{22} &= 1/3 \end{aligned} \quad \ldots\ldots 34.$$

Along the principal axis, the only inefficiency is a factor of 2 on shear performance, which will result in ε_s' being half of ε_s. This form of structuring is the most efficient means of diluting material in one direction. Timber and all prismatic members share these performance characteristics to some extent, but cross grain properties are sensitive to curvatures and discontinuities in the sectional geometry.

The compressive strength of such a structure may be governed by instability in its members. A long panel of width L and thickness t, with its long sides simply supported, has a critical compressive strain, when loaded lengthways of (Ref 8):

$$\varepsilon_c = \frac{\pi^2}{3(1-\nu^2)} \frac{t^2}{L^2} \quad \ldots\ldots 35$$

which is dependent on the slenderness λ defined as:

$$\lambda = L/t. \quad \ldots\ldots 36$$

Slenderness itself is dependent on m. Fig 5 shows several regular tessellations. In order to achieve the maximum possible ε_c for a given m, no member should be more slender than any other. Assuming each has equal slenderness, λ can be shown to be inversely proportional to m. The value of λm for each tessellation is given in the figure. Honeycomb offers the lowest λm, and so offers the highest ε_c for a given material concentration.

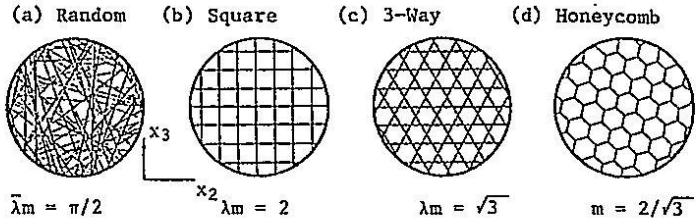


Fig 5

For the random tessellation the $\bar{\lambda}$ in $\bar{\lambda}$m is a mean slenderness and inevitably there must be scatter about the mean. However, $\bar{\lambda}$ is not much smaller than the λ for the three way lay-up which indicates that it is likely to be a lower bound of λ for this type of continuous tessellation.

A slice of this structuring, w deep in the x_1 direction, represents a panel of lineal members. Apart from the random orientation of its members, this is typical of a panel of bracing. When w is much smaller than $\bar{\lambda}$, E_{22} is the only material property that affects the performance of the panel and it can be seen from Eqn 33 that:

$$\begin{aligned} \eta_{22} &= 1/3 \\ \nu_{23}' &= 1/3 \end{aligned} \quad \ldots\ldots 37$$

Instability of this panel may be governed by λ. Eqn 24 may be rewritten as:

$$\varepsilon_c = \pi^2/12\lambda^2 \quad \ldots\ldots 38$$

for a rectangular sectioned member, ignoring shear flexibility.

The structuring just described may be considered as a material perforation; either as a deep perforation of a volume or as a shallow perforation of a surface. In both cases, the effects are approximately described by the following equations:

$$\begin{aligned} &\text{Efficiency} && \eta = 1 \text{ or } 1/3 \\ &\text{Slenderness} && \lambda = \tfrac{1}{2}\pi/m \\ &\text{Buckling strain} && \varepsilon_c = 2m^2 \text{ or } \tfrac{1}{2}m^2 \end{aligned} \quad \ldots\ldots 39$$

A dilute, low density structure is achieved when m is small However, it is likely to have instability problems accordin to the last equation. Suppose a material concentration factor of 1/100 were sought, but the corresponding buckling strain of approximately 0.02% (Eqn 39) were unacceptable. A possible solution would be to dilute in two stages, each time by a factor of 1/10; the two stages are on different scales, one stage structuring the material out of which the other stage is made. A detail of both stages together is shown in Fig 6. The buckling strain would then be calculated by Eqn 39 as 2%.

This example introduces the recursive nature of the structuring process. Since structural material has the same type of properties as material, structured material can itself be structured. Indeed, structures can be structured, and the divide between material and structure is not sharply defined. This process is not confined to two stages of

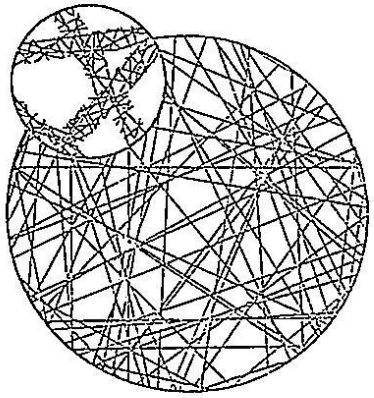


Fig 6

structuring. If Eqn 39 defines the process for one stage, for n stages:

$$\text{Material concentration factor, } m_n = m^n$$
$$\text{Efficiency, } \eta_m = 1 \text{ or } 1/3^n \qquad \ldots\ldots 40.$$

λ and ε_c are as defined by Eqn 39 for each stage of structuring. Notice that the efficiency can stay as unity for prismatic structuring, but the price to pay is in the geometrical intricacy and thinness of material.

The structured material just considered is not isotropic. When the planes of material are oriented completely randomly, the resulting structure is isotropic and may be shown to have the following properties:

$$E' = \tfrac{1}{3}mE$$
$$\nu' = 1/5 \qquad \ldots\ldots 41.$$

Each plane is covered with random lines of intersection, so would look like Fig 5a. It can be shown that the mean slenderness $\overline{\lambda}$, based on the mean side length of all panels divided by the common thickness t, is again given by:

$$\overline{\lambda} = \tfrac{1}{2}\pi/m \qquad \ldots\ldots 42.$$

which is equal to $\overline{\lambda}$ for prismatic perforation, as given in Fig 5a. All planes of material can therefore be perforated, as a further stage of structuring, each plane being left with m times its original material. The structure then becomes a random lattice of members with thickness t and mean slenderness given by Eqn 42.

The material concentration factor for this second stage is $\frac{1}{3}m$, not m, due to the fact that every lattice member lies on the intersection of two planes. Hence, another stage of dilution has been achieved, without a change in slenderness or a drop in buckling stress, and without increased geometric intricacy or reduction in thickness. The other properties of this isotropic lattice are:

$$E' = mE/6$$
$$\nu' = 1/4 \qquad \ldots\ldots 43.$$

The effects of isotropic latticing are approximately described by the following equations.

$$\text{Efficiency} = 1/6$$
$$\text{Slenderness} \simeq \pi/2\sqrt{2m}$$
$$\text{Buckling strain } \varepsilon_c = 2m/3 \qquad \ldots\ldots 44.$$

Finally, the effect of structural shear on member flexure will be considered. Fig 7 shows a node in a structure with members spanning in directions OX, OY and OA. If all nodes are identical and the structure is under uniform strain, the nodes assume displacements which are compatible with the strain of the structure regarded as a material. Figure 7 shows the extreme case of maximum shear strain γ coinciding with the orientation of members OX and OY. When members OX are rigid, members OY suffer the maximum end rotations possible, which is γ.

It can be shown that the flexural strain in a prismatic member subjected to equal end rotations γ is always much less than γ, for all slenderness ratios and for all axial strains.

The peak shear strain is never greater than twice the peak

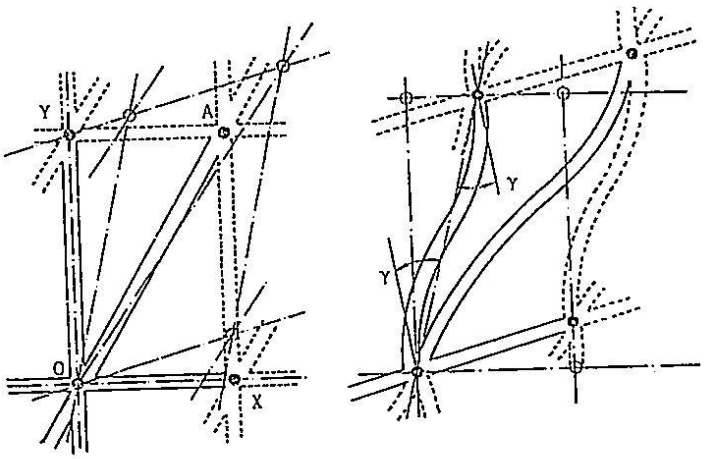


Fig 7

direct strain. Hence, in triangulated structures with prismatic members, member flexure is never a dominant effect when it is caused solely by structural shear.

This section has demonstrated how spatial and surface structuring can be described in terms of continuum mechanics without reference to any boundaries which would identify structures. Structuring is simply related to patterns and to their proportions. It is recursive; structuring can itself be structured, and the utilisation of high strength material in diluted form is likely to require more than one stage of structuring. One stage of structuring results in lineal dispersion of material about a surface, as might be achieved by perforating a solid surface. It also results in surface dispersion of material in space. If these surfaces are themselves perforated in a second stage of structuring, then lineal dispersion in space, ie latticing, is achieved.

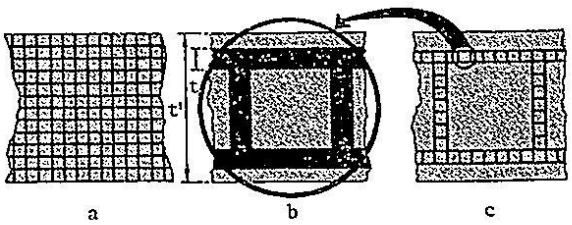


Fig 8

STRUCTURAL PERFORMANCE

The last section dealt with material dilution without reference to structure boundaries, and consequently only considered multiple cell, not single cell, structuring. An example of multiple cell structuring is shown in Fig 8a. The shaded envelope is its representative solid plate. Fig 8b shows the single cell structure which shares the same representative material and solid plate thickness; A and I are the same for both, and the slendernesses of the members and ε_c are almost identical. The difference is one of scale; the multiple cell section has thinner members and many more of them. The greater intricacy of multiple cell structuring is the reason why single cells are usually chosen to cover the shortest dimension of a structure.

Fig 8c, which shows single cell second order structuring, has been drawn to illustrate the difference between multiple cell and higher order structuring. The second order structure is intricate, but has a far smaller A and I than the other two. ε_c is unchanged, but dilution has been achieved. The multiple cells of Fig 8a do not contribute to dilution any more than the single cell of Fig 8b.

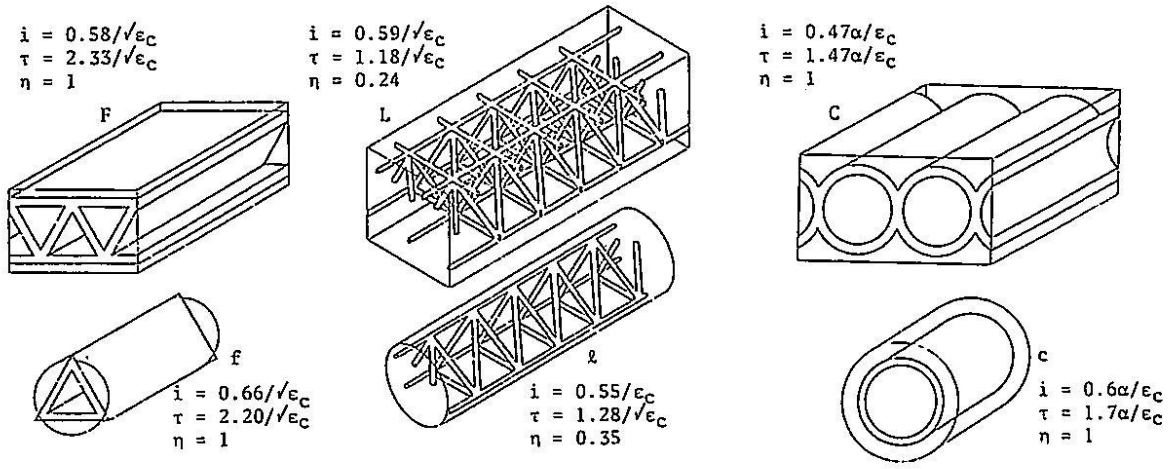


Fig 9

Two material thicknesses are relevant to each stage of structuring; that of the representative material and that of what will be referred to as the "parent" material. In the first stage of structuring, the parent material is actual material, whilst in subsequent stages it is the representative material of the previous stage. The thickness t' of the representative material is always much bigger than the thickness t of the parent material, and a thickness ratio τ will be defined as:

$$\tau = t'/t \qquad \ldots\ldots 45.$$

The effect of curvature on structural performance is generally adverse, reducing stiffness and strength by introducing bending and torsion into members. Curved shells however, can offer exceptional performance. Cylindrical shells are locally more stable than prismatic members made out of flat plate. Ref 8 gives the local buckling strain ε_c of a cylinder of radius R and thickness t as:

$$\varepsilon_c = t/R\sqrt{3(1-\nu^2)} \qquad \ldots\ldots 46.$$

In practice, the buckling strain is considerably less than indicated when t/R is very small, Ref 11. This loss of buckling performance may be described by multiplying the right hand side of Eqn 46 by α. α may not be much smaller than unity at high strains, and falls to around 0.1 at buckling strains of 10^{-5}, Ref 11.

All unit cells can be linked to form a line or a surface, depending on whether cells are linked in one or two directions. Linking in three directions would form a solid. The laterial linking of cylinders does present practical and theoretical difficulties, but so as not to make an exception of this structural form, it will be assumed that a cylinder may be joined across a diameter to two adjacent cylinders to produce surface structuring.

Three types of cell are illustrated in Fig 9, which represent typical ways of structuring flat plate, beams and curved plate. Each type of structuring can generate surfaces or lines of hierarchical structuring. It is convenient to assign a letter to a particular cell geometry and to use upper and lower case to differentiate between surface and line generation. Accordingly, the letters F, f, L, ℓ, C and c have been designated to the three cell types of Fig 9.

The F cell structure is prismatic, made up of flat equilateral plates. The lattice cell, L, is made up of members of two different lengths all having the same diameter. Members parallel to the surface in L have the same length. Diagonal members are slightly longer. The C cell is a cylindrical tube. Beneath the illustrations the properties of structuring, i, τ and η, are given in terms of ε_c. These have been derived by the techniques already described.

We are now in a position to calculate i, τ and η in terms of ε_c for any structuring using F, L or C and any higher order combinations of them. The effect of combining them hierarchically is found by multiplying together the contributions of each of the stages. For example, if material is formed into a double-skinned deck structure (F), which is then wrapped into a cylinder (c) and a number of these cylinders are formed into a lattice beam (ℓ), and the slenderness of members in each stage is such that ε_c is the same for each, then:

$$\begin{aligned} &\text{Dilution factor, } i = i_\ell i_c i_F \\ &\text{Thickness ratio, } \tau = \tau_\ell \tau_c \tau_F \\ &\text{Efficiency, } \eta = \eta_\ell \eta_c \eta_F \end{aligned} \qquad \ldots\ldots 47.$$

The resulting structural properties are given by:

$$\begin{aligned} E' &= E/i \\ t' &= t/\tau \\ \rho' &= \rho/\eta i \end{aligned} \qquad \ldots\ldots 48.$$

and ε_c is the local buckling strain in the principal direction throughout the structure. Conceptually, we have at our disposal a continuous range of materials of known pro-

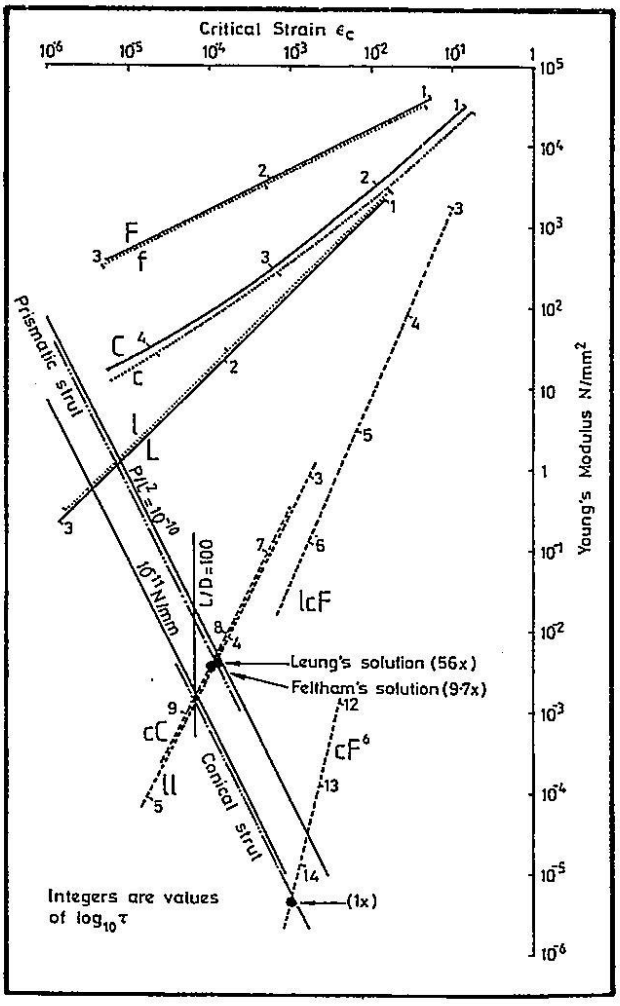


Fig 10

perties. Choosing an appropriate structural form is akin to material selection.

Fig 10 illustrates the structuring process. Critical strain ε_c is measured horizontally, and Young's modulus E' of the representative solid material is measured vertically both to log scales. For convenience, an actual material will be considered which has an E of 10^5 N/mm^2, making the top of the graph correspond to i = 1, and E' = 10^4 correspond to i = 10 etc. Using the expressions for i in Fig 9, six lines can be drawn. The integers are values of $\log_{10}\tau$ and represent the degree of intricacy of the structuring. Notice that there is very little to distinguish the upper and lower case lines from each other. In the absence of all imperfection, the lines C and c practically coincide with L and ℓ. The curving of C and c is the result of choosing realistic values of α.

The line corresponding to the example of structuring in the order ℓcF (Eqn 47) is also shown. Because of the log scale it can be constructed by suspending the ℓ, c and F lines consecutively down from the strain axis.

Establishing that conceptually there is a full spectrum of materials has been an analytical process. The design process is different as a result; the designer has a wider choice of materials for the design of a solid structure. For example, consider designing a strut of length L to carry a force P. If the structure loading coefficient, P/L^2 (Ref 12) is small, then the strut is likely to be highly structured and overall buckling will be a principal parameter. If the strut is a solid cylinder of diameter D:

$$P = \pi D^2 E' \varepsilon_c / 4 \qquad \ldots\ldots 49.$$

If ε_c is given by Eqn 25, ignoring the effect of shear flexibility, then eliminating D from Eqn 49:

$$P/L^2 = 4E'\varepsilon_c/\pi \qquad \ldots\ldots 50.$$

This equation defines a family of diagonal straight line contours on Fig 10, two of which are illustrated. Where any such line cuts proposed structured material lines, there is a "just stable" solution. There will always be a choice between simple structures, working at lower strains using more material and more intricate structures using less material working at higher strains. There may of course be a stiffness criterion in the design that overrides that of strength.

Reproduced on this page is the Strut Problem. It was first published in "New Civil Engineer" (Ref 13) beside an article which introduced the concept of structuring as a process of material dilution. 20 entries were received from a readership of nearly 50,000.

THE STRUT PROBLEM

Deep in space where there is no extraneous influence, two equal and equally charged 10m diameter spheres are prevented from moving apart by a 1mm diameter wire, which in this situation is stressed to its working load. The wire is made of 'materium', a hypothetical uniform material. Its original length of 1km is extended under load by 1m.

Now assume the charge of one sphere is reversed. How many times more materium would be the minimum needed to form a strut to keep the spheres apart and what form of structure would it take if:

(a) the minimum manufacturable thickness of materium were 1mm?

(b) there is no lower limit of thickness?

Assume materium has no manufacturing constraints. Any shape and size can be fabricated free of imperfections. Its elastic properties and limiting stress are the same in compression and tension.

The strut must be designed to be stable.

The design criterion for the Strut Problem is effectively given by a structure loading coefficient of 7.85×10^{-11} N/mm^2 on Fig 10, labelled "prismatic strut", and a target strain of 10^{-3}. D. Leung won the first part of the problem, by designing 1mm wire into a latticed lattice ℓℓ structure. I. Feltham produced the winning solution to the second part, with the lightest valid entry. He considered the wall of his cylindrical tube to be made up of tubular straws. At 9.7 times as heavy as the tie, his was the lightest valid solution. He was the only entrant to utilise material dilution in his calculations.

Both solutions are marked on Fig 10; Leung's as ℓℓ and Feltham's as cC. As permitted by the conditions of the problem, α has been taken as one, which is why cC is straight and much further down than would be anticipated from the positions of the curves c and C above. Both struts have diameters close to 10m, the sphere diameter. The vertical line "L/D = 100" indicates that to the left of this line prismatic struts are satisfactory. To the right of it some tapering will begin to be necessary, and eventually a fully tapered strut will be required, which will need extra dilution at midspan. The design line for a conical strut with solid 1mm diameter ends is illustrated.

Had Feltham gone to two further orders of structuring, cC^3, he would have achieved a fully stressed strut which used no more material than the tie, a factor of one solution. The author's factor of one solution was cF^6, and for the first part he proposed a stayed strut, being an unstressed 1mm wire framework staying the fully stressed 1mm tie wire in compression. Freed from compressive strain, the stays do not themselves need staying, and with the greater freedom of structural form admitted, a strut using only 30 times more material was possible, compared with Leung's factor of 56. All solutions are illustrated in Ref 14.

CONCLUSION

This paper has demonstrated that structuring is a process of material dilution; that there is no clear divide between structure and material, and that the handling of structured material is an analytical process, not a design process. In introducing these ideas, the aim has been to lay a firm theoretical foundation and not to draw any practical conclusions.

ACKNOWLEDGEMENTS

The author wishes to thank his wife Nina, his colleagues at Atkins Research and Development, Dr. John Parkinson and the editorial staff of New Civil Engineer, and Sir Alan Harris for his advice and encouragement.

REFERENCES

1. R K LIVESLEY, Matrix Methods of Structural Analysis, 2nd ed., Pergamon Press, 1975
2. O C ZIENKIEWICZ, The Finite Element Method, 3rd ed., McGraw-Hill, 1977.
3. A HRENNIKOFF, Solution of Problems of Elasticity by the Framework Method, J. of Appl. Mech., Dec. 1941.
4. D McHENRY, A Lattice Analogy for the Solution of Stress Problems, J. of Inst. Civ. Eng., Dec. 1943.
5. E C HAMBLY, Bridge Deck Behaviour, Chapman and Hall, 1976.
6. Z S MAKOWSKI, Ed. Analysis, Design and Construction of Double-Layer Grids, Applied Science Publishers, 1981.
7. M A BIOT, Increase of Torsional Stiffness of a Prismatic Bar due to Axial Tension, J. of Appl. Phys., Vol. 10, 1939.
8. S P TIMOSHENKO and J M GERE, Theory of Elastic Stability, 2nd Ed., McGraw-Hill, 1961.
9. R L BISPLINGHOFF, J W MAR and T H H PIAN, Statics of Deformable Solids, Addison-Wesley, 1965.
10. A H NAYFEH and M S HEFZY, Effective Constitutive Relations for Large Repetitive Frame-Like Structures, Int. J. Solids & Structures, Vol. 18, No. 11, 1982

11. C G FOSTER, On the Buckling of Thin-Walled Cylinders Loaded in Axial Compression, J. of Strain Analysis, Vol. 16, No. 3, 1981.

12. H L COX, The Design of Structures of Least Weight, Pergamon Press, 1965.

13. J PARKINSON, Structures Analysed Anew, New Civil Engineer, No. 513, 21st Oct. 1982.

14. J PARKINSON, Ultimate Strut Solution, New Civil Engineer, No. 521, 23/30 Dec. 1982.