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STRUCTURAL SIMPLICITY THROUGH A LENTICULAR PATTERN

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Abstract—A two-dimensional lenticular pattern is introduced which can describe a family of structures for which deformations, equilibrium and material properties have one-dimensional simplicity. This simplicity permits large strains and non-linear material performance to be analyzed very simply. Following a description of the most general pattern, the simplest is described, which is a two-flanged structure as simple to analyze as Shanley's model column. The axial force/displacement response derived from an analysis can be expressed as a stress/strain curve, which is a description of material performance. The analysis is then an example of a material transformation produced by a shape: the response curve is a function derived from another function, the stress/strain curve representing the constituent material's performance, by a transformation determined solely by the shape. The performance of an initially straight strut made of elastoplastic material is explored and its stability is discussed.

NOTATION

- A total sectional area of member
- a semi-length
- a_0 initial semi-length
- b semi-width of member
- E Young modulus, $d\sigma/d\varepsilon$
- F material flexibility, $d\varepsilon/d\sigma$
- f transverse line loading on member
- f_1, f_2, f_β transverse forces per unit area
- I inertia of midspan section of member
- L initial length of member
- P force
- p lenticular profile function
- r local scale factor
- s arc length of part of a lenticular curve
- s_a semi arc length of whole of a lenticular curve
- t state parameter
- u longitudinal shortening of member
- v transverse displacement of member
- W work
- x, y distances
- α, β dimensionless coordinates
- ε strain
- ε_0 initial out-of-straightness of member
- ε_E stockiness of member, also its elastic buckling strain
- Φ cumulative normal distribution function
- ϕ material stress/strain function
- ψ structural stress/strain function
- κ curvature, midspan curvature
- μ, ν material identifiers
- σ stress

Subscripts

- A relating to overall scale
- C critical or maximum value
- E associated with elastic buckling
- L longitudinal
- M material
- T transverse

0 at midspan ($x = 0$)
1, 2 in flanges 1 or 2
α, β in α- or β-direction
μ, ν in μ- or ν-direction

1. INTRODUCTION

A structural member suffers static failure and becomes unstable when its internal state is no longer governed entirely by small changes in the positions of its supports, the applied loading and its material properties. The onset of static failure can be described mathematically as a singularity in the equations governing the static behaviour of a mathematical model of the member. This is an instability. It is followed by a period of dynamic behaviour when time and inertial effects augment the number of equations and variables in the model. During this time the failing member will dictate the loading in the surrounding structure, as its contribution to supporting the surrounding structure's internal loading rapidly reduces. Dissipation of energy will bring the dynamic period to a close, when again the member's behaviour is governed statically. This failure of the member will not necessarily have caused failure of the surrounding structure, but it may have done: reduction of its load may have caused other members to be sufficiently highly loaded to suffer static failure, and a region of failure could have spread within the surrounding structure to the extent of failing part or all of it. What governs structural failure is not member strength, but lack of member stiffness. As an example of this, consider a single prismatic column to be a whole structure, and consider its material as its "members": according to Euler's column-buckling formula, the strength of the column (provided it is not too stocky) is $\pi^2 EI/L^2$, an expression that does not contain its material's strength, only its stiffness, E.

The axial load/axial displacement response of a prismatic member is not simply calculable when the stress/strain curve of its material is non-linear, even using numerical techniques [1]. Calladine [2] has described the history of column-buckling theories up to Shanley's demonstration [3] in 1947 that Euler's formula applied to ductile alloys if the materal stiffness, E, were taken as the tangent modulus of the yielding alloy. For 50 years previously it had been believed that a greater value of E should have been used to account for the material being stiffer when it was being unloaded than when it was being loaded. Shanley's argument relied on the model column shown in Fig. 1: its infinitely rigid arms transmit the column load through two miniature columns of material to give just two material stresses in each load/deformation state, simplicity he needed for his demonstration.

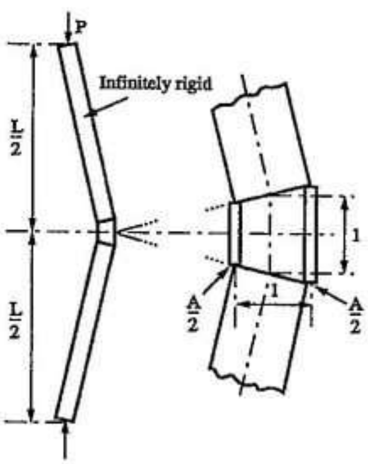

FIG. 1. Shanley's model column is a short two-flanged cell sandwiched between long infinitely rigid arms [3].

This paper shows that this same simplicity belongs to a two-flanged lenticular structure, free of the artificiality of the rigid arms. As will be shown, it is particularly amenable to non-linear analysis, and load/deflection responses can be derived from material stress/strain curves of arbitrary shape rather easily. It is the simplest of a family of model structures based on a lenticular-shaped coordinate system, and the basis for the whole family is described first. The peculiar one-dimensional simplicity of the structural system permits the consideration of large strains, and the next section introduces large deformations in a simple one-dimensional system in preparation for the more elaborate application of the same principles to the two-dimensional system that follows.

2. ONE-DIMENSIONAL LARGE DEFORMATIONS

Initially, in state $t = 0$, a straight continuous bar is $2a_0$ long and all its material is identified by all the real numbers in the interval -1 to 1 according to the rule

$$\mu = \frac{x}{a_0}$$

where x is the distance of any point in the material from the centre of the bar at this instant. After subsequent continuous deformation to state t, x is the distance of the point μ from the point that has become the centre of the bar, and α is defined as

$$\alpha = \frac{x}{a}$$

where $2a$ is the length of the member in this state. a is a function of t only, while α is a function of μ and t, i.e.

$$a = a(t)$$

$$\alpha = \alpha(\mu, t) \quad -1 < \alpha < 1.$$

These functions are continuous, and because of the way they have been defined

$$\alpha(\pm 1, t) = \pm 1$$

$$\alpha(\mu, 0) = \mu.$$

The piece of material $d\mu$, originally $a_0 d\mu$ long, has become $a d\alpha$ long, and a local scale factor, r, is defined as

$$r = \frac{a}{a_0}\frac{\partial \alpha}{\partial \mu}$$

so that the length $a d\alpha$ may be expressed as $a_0 r d\mu$. A state change of dt produces a change in length

$$= a_0 \frac{\partial r}{\partial t} d\mu \, dt.$$

Hence an incremental strain, $d\varepsilon$, defined as the increase in length divided by the total length, is given by

$$d\varepsilon = \frac{1}{r}\frac{\partial r}{\partial t} dt = \frac{1}{r} dr.$$

Integrating from state $t = 0$ to state t, we derive a true strain, ε, where

$$\varepsilon = \ln r.$$

3. A LENTICULAR COORDINATE SYSTEM

3.1. *Geometry*

Figure 2 shows two sets of curves, one of which is a lenticular set defined by

$$y \frac{d^2 y}{dx^2} = -\frac{\pi}{2}\beta^2 \tag{1}$$

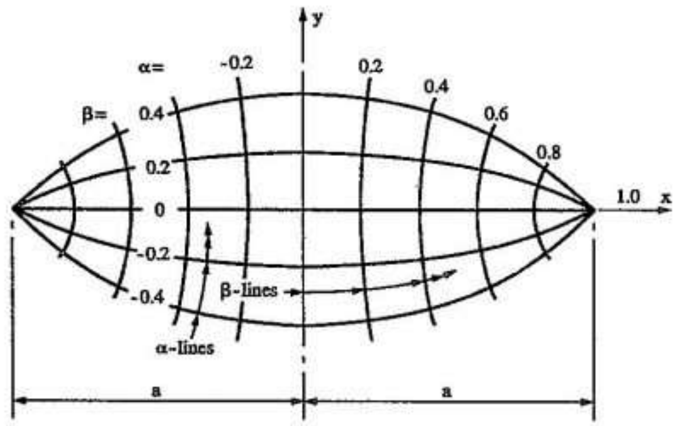

FIG. 2. The lenticular pattern formed by the α-lines is defined by Eqn (1).

scaled and positioned so that $y = 0$ when $x = \pm a$. The parameter β distinguishes each curve of this set. It is shown in the Appendix that

$$\frac{x}{a} = \pm\left[2\Phi\left(\sqrt{2\ln\frac{\beta a}{y}}\right) - 1\right]$$

where

$$\Phi(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u} e^{-x^2/2}\,dx.$$

A profile function, p, which describes these curves is defined by

$$y = \beta a p(x/a). \tag{2}$$

Introducing primes to denote successive derivatives, Eqn (1) implies

$$p(x/a)p''(x/a) = -\pi/2. \tag{3}$$

For small dy/dx, the arc length s along one of these curves, from where it cuts the y-axis, is given by

$$\begin{aligned} s &\approx \int_0^x 1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2 dx \\ &= x + \int_0^x \beta^2 (p'(x/a))^2\,dx \\ &= x\left(1 + \frac{\pi}{4}\beta^2\right) + \frac{1}{2}a\beta^2 p(x/a)p'(x/a). \end{aligned} \tag{4}$$

As x approaches a the last term above approaches zero, and for small β the semi arc length s_a becomes

$$s_a \approx a\left(1 + \frac{\pi}{4}\beta^2\right). \tag{5}$$

Equation (4) relies on dy/dx being small, so it cannot be argued that Eqn (5) follows from it since when x approaches a, dy/dx approaches infinity. A rigorous derivation of Eqn (5) appears in the Appendix.

The dimensionless parameter α is defined as

$$\alpha = s/s_a, \tag{6}$$

which will be used as the parameter for each lenticular curve. These curves will be referred to as α-lines. From Eqns (4) and (6), remembering that β is small,

$$x/a = \alpha - \tfrac{1}{2}\beta^2 p(\alpha)p'(\alpha). \tag{7}$$

Equations (7) and (2) give x/a and y/a in terms of α and β, and define the coordinate system of Fig. 2. One property of this system is that the nine β-lines shown dissect each α-line into 10 parts of equal arc length. Another property is that for small β and away from the ends, these two sets of lines are orthogonal. From Eqns (7) and (2)

$$\frac{\partial x}{\partial \beta} = -\beta a p(\alpha)p'(\alpha)$$

$$\frac{\partial y}{\partial \beta} = a p(\alpha).$$

Hence

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{\alpha=\mathrm{const}} = -\frac{1}{\beta p'(\alpha)}.$$

This is the slope of a β-line. The slope of an α-line from Eqn (2) is $\beta p'(\alpha)$, so their orthogonality is demonstrated.

A third property of this coordinate system is that, for small β, the curvature of any α-line is inversely proportional to y. This follows from Eqn (1). These three properties permit a two-dimensional system of stresses and strains to have practically one-dimensional simplicity, as will be shown below.

3.2. *Continuity of deformation*

Initially, in state $t = 0$, a patch of material bounded by two α-lines is $2a_0$ long and all its material is identified by all the real dyads (μ, ν), where μ and ν are the α and β coordinates of each point of the material at this instant. After subsequent continuous deformation to state t, the (α, β) coordinates may again be measured for each point (μ, ν) in the material. The initial patch may have grown to be $2a$ long and the coordinate system will have grown to match. a is a function of t only, and for a general deformation both α and β could be independent functions of μ, ν and t. However, only deformations of the following form will be permitted:

$$\begin{aligned} a &= a(t) \\ \alpha &= \mu \\ \beta &= \beta(\nu, t). \end{aligned} \tag{8}$$

The first of these equations permits the global scale to change freely, the second constrains all material to move along its allocated β-lines: a point identified by μ must always have that as its α-coordinate. Finally, a strip of material that was initially bounded by two α-lines maintains its characteristic shape by always being bounded by two α-lines, but not necessarily the initial two. By definition

$$a(0) = a_0$$

$$\beta(\nu, 0) = \nu.$$

The dissection property ensures that as the infinitesimally wide strip bounded by ν and $\nu + \mathrm{d}\nu$ moves, its elongation is spread equally along its length so that the strip is uniformly strained in any state t. If r_α is the ratio of its current length to its original length, then from Eqn (5)

$$r_\alpha = \frac{a}{a_0}\left(\frac{1 + \frac{\pi}{4}\beta^2}{1 + \frac{\pi}{4}\nu^2}\right)$$

$$\approx \frac{a}{a_0}\left(1 + \frac{\pi}{4}(\beta^2 - \nu^2)\right)$$

and

$$\varepsilon_\alpha = \varepsilon_A + \frac{\pi}{4}(\beta^2 - \nu^2) \tag{9}$$

where $\varepsilon_A = \ln(a/a_0)$. A strip bounded by two β-lines stays bounded by the same two β-lines but is not necessarily uniformly strained:

$$r_\beta = \frac{a}{a_0}\frac{\partial \beta}{\partial \nu}$$

and

$$\varepsilon_\beta = \varepsilon_A + \ln\frac{\partial \beta}{\partial \nu}. \tag{10}$$

There are no shearing strains, $\varepsilon_{\alpha\beta}$, because the μ- and ν-lines always remain orthogonal.

3.3. *Force equilibrium*

Stresses must be of the following form:

$$\sigma_\alpha = \frac{\sigma_{\alpha 0}(\beta)}{p(\alpha)} \tag{11}$$

$$\sigma_\beta = \frac{\sigma_{\beta 0}(\beta)}{p(\alpha)} \tag{12}$$

$$\sigma_{\alpha\beta} = 0 \tag{13}$$

where $\sigma_{\alpha 0}$ and $\sigma_{\beta 0}$ are the values of σ_α and σ_β at midspan ($\alpha = 0$). Both may be arbitrary functions of β only, and $p(\alpha)$ is the profile function defined by Eqn (2).

Since the width of an infinitesimally wide strip bounded by β and $\beta + \mathrm{d}\beta$ is $ap(\alpha)\mathrm{d}\beta$, the tangential force in the strip is $\mathrm{d}P$, where

$$\mathrm{d}P = a\sigma_{\alpha 0}(\beta)\,\mathrm{d}\beta,$$

which is independent of α. The curvature of this strip is κ, which, from Eqns (2) and (3), is given by

$$\begin{aligned}\kappa &= \frac{\beta}{a}p''(\alpha)\\ &= -\frac{\pi}{2}\frac{\beta}{ap(\alpha)}.\end{aligned} \tag{14}$$

Suppose a force per unit area, i.e. a two-dimensional body force, is applied to the material in the β-direction, in order to keep each lenticular strip in equilibrium with its internal tangential force $\mathrm{d}P$, this body force f_1 acts in the β-direction and has magnitude

$$f_1 = \frac{\pi}{2}\beta\frac{\sigma_{\alpha 0}(\beta)}{a(p(\alpha))^2}.$$

σ_β can be equilibrated by a body force f_2 acting in the same direction, where

$$\begin{aligned}f_2 &= -\frac{1}{ap(\alpha)}\frac{\partial \sigma_\beta}{\partial \beta}\\ &= -\frac{\sigma'_{\beta 0}(\beta)}{a(p(\alpha))^2}.\end{aligned}$$

Hence stresses distributed according to Eqns (11), (12) and (13) can be equilibrated by an axial force P given by

$$P = \int_{\beta_1}^{\beta_2} a\sigma_{\alpha 0}(\beta)\,\mathrm{d}\beta$$

and a body force in the β-direction of f_β where

$$f_\beta = f_1 + f_2$$
$$= \frac{\frac{1}{2}\pi\beta\sigma_{\alpha 0}(\beta) - \sigma'_{\beta 0}(\beta)}{a(p(\alpha))^2}. \tag{15}$$

When f_β is zero, a simple relationship exists between $\sigma_{\alpha 0}$ and $\sigma'_{\beta 0}$.

3.4. *Material properties*

Material strains in response to stress. Its response can be expressed by its flexibility, F, to small stress increments, defined as

$$F = \frac{d\varepsilon}{d\sigma}.$$

At every point in the material being considered there are two stresses and two strains giving four flexibilities, $F_{\mu\mu}$, $F_{\mu\nu}$, $F_{\nu\mu}$ and $F_{\nu\nu}$, where

$$F_{\mu\nu} = \frac{\partial \varepsilon_\mu}{\partial \sigma_\nu}.$$

Each of these four flexibilities must be of the form

$$F_{\mu\nu} = p(\mu) F_{\mu\nu 0}(\nu, t), \tag{16}$$

so that any stress increment compatible with Eqns (11) and (12) will result in a deformation increment compatible with Eqns (9) and (10).

3.5. *Structural analysis*

The importance of this system is that the principal unknowns are functions only of ν or β, not of μ or α, effectively one-dimensionalising a two-dimensional problem. For example, stress distributions need to be calculated across the system, but not along it, since longitudinal distributions all follow the standard pattern of intensities that are inversely proportional to $p(\alpha)$. For any state step, dt, from a known state, t, when any two of

the deformation increment
the load increment
the material properties

are known, the other one can be calculated.

Provided the material density varies with μ inversely proportionally to $(p(\mu))^2$, transverse inertial effects also become one-dimensional.

4. A TWO-FLANGED STRUCTURE

A particularly simple and interesting structure is obtained by selecting three lenticular strips in the α-direction, two infinitesimally thin ones defined by the parameters ν_1 and ν_2, and the third filling the gap between them. The outer two strips are two flanges, 1 and 2, of the same thickness, $a d\nu$, and made of the same material. The inner strip is a web connecting the flanges, and this is made of an ideal material, perfectly stiff in the ν-direction and perfectly flexible in the μ-direction, which keeps the flanges a constant distance apart but does not otherwise participate in transmitting axial load: the two flanges carry all of it. A three-dimensional structure is achieved if the flanges are considered as plates. If they have constant width and constant thickness, the material properties no longer need to vary with μ according to $p(\mu)$ as stated in Eqn (16): the constant forces P_1 and P_2 in the two flanges are now compatible with their two uniform strains when the material in the uniformly thick flanges is itself uniform.

An elevation of this structure is shown in Fig. 3. In any given state, flange 1 is stressed and strained throughout its length by σ_1 and ε_1 and flange 2 by σ_2 and ε_2. For the remainder of the paper, strains will be considered to be very small, and compression will be taken as positive.

Suppose an initially straight structure, of length L and breadth $2b$ and constant sectional area A (of the two flanges together), is deformed according to the previous constraints so that it reduces in length by u and its midpoint deflects transversely by v_0, as shown in Fig. 4. The lengths of the flanges can be calculated from Eqn (5). Calculation of them before and after deformation gives

$$\varepsilon_1 = \frac{u}{L} + \frac{\pi}{L^2}(b^2 - (v_0 - b)^2)$$

$$\varepsilon_2 = \frac{u}{L} + \frac{\pi}{L^2}(b^2 - (v_0 + b)^2).$$

The displacements, u and v_0, will be expressed as longitudinal and transverse "structural strains", ε_L and ε_T, using the following substitutions:

$$\varepsilon_L = \frac{u}{L}$$

$$\varepsilon_T = \frac{2\pi b}{L^2} v_0. \tag{17}$$

Then

$$\varepsilon_1 = \varepsilon_L + \varepsilon_T - \frac{L^2}{4\pi b^2}\varepsilon_T^2 \tag{18}$$

$$\varepsilon_2 = \varepsilon_L - \varepsilon_T - \frac{L^2}{4\pi b^2}\varepsilon_T^2. \tag{19}$$

A factor, ε_E, is defined as

$$\varepsilon_E = \frac{2\pi b^2}{L^2}.$$

This is a measure of the stockiness of the member, and it will be shown to be also the elastic buckling strain. Equations (18) and (19) can be rearranged to give

$$\varepsilon_T = \tfrac{1}{2}(\varepsilon_1 - \varepsilon_2) \tag{20}$$

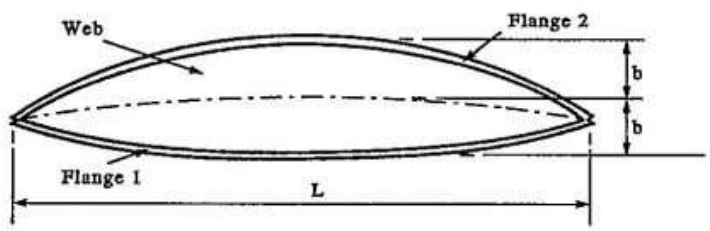

FIG. 3. A bent two-flanged model structure based on the lenticular pattern.

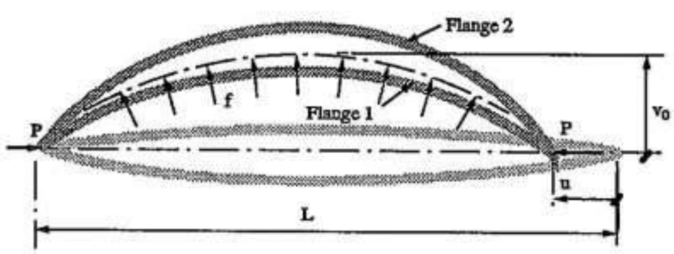

FIG. 4. Deformations u and v_0 of a two- flanged model structure under a transverse load f per unit length and an axial load P.

$$\varepsilon_L = \frac{1}{2}(\varepsilon_1 + \varepsilon_2) + \frac{\varepsilon_T^2}{\varepsilon_E}. \tag{21}$$

The applied loading is an axial compressive force, P, and a transverse distributed load of the form described by Eqn (15). Expressed as a line loading of intensity f per unit length

$$f = \frac{f_0}{p(\alpha)},$$

where f_0 is the intensity of f at midspan. Displacements, v, vary according to

$$v = v_0 p(\alpha)$$

so the work done, dW, in moving f through dv over the length L is

$$dW = Lf_0 dv_0.$$

Longitudinal and transverse "structural stresses", σ_L and σ_T, will complement the structural strains. Just as ε_T is being used instead of v, so σ_T will be used instead of f, and chosen to be compatible with ε_T for energy calculations when applied to the whole volume of flange material, AL. Then

$$Lf_0 dv_0 = AL\sigma_T d\varepsilon_T$$

and, by Eqn (17),

$$\sigma_T = \frac{L^2}{2\pi Ab} f_0. \tag{22}$$

Also, let

$$\sigma_L = \frac{P}{A}.$$

Axial components of f will be neglected. Resolving longitudinally:

$$P = P_1 + P_2.$$

Introducing the material stresses $\sigma_1 = P_1/\frac{1}{2}A$ and $\sigma_2 = P_2/\frac{1}{2}A$, then

$$\sigma_L = \tfrac{1}{2}(\sigma_1 + \sigma_2). \tag{23}$$

Equation (14) gives the midspan curvatures of the flanges κ_1 and κ_2 as

$$\kappa_1 = -\frac{2\pi}{L^2}(v_0 - b)$$

$$\kappa_2 = -\frac{2\pi}{L^2}(v_0 + b).$$

Resolving transversely at midspan

$$f_0 = -\frac{2\pi}{L^2}(P_1(v_0 - b) + P_2(v_0 + b)).$$

Therefore, substituting σ_T for f_0 using Eqn (22),

$$\sigma_T = \frac{1}{2}(\sigma_1 - \sigma_2) - \frac{\varepsilon_T}{\varepsilon_E}\sigma_L. \tag{24}$$

Equations (20) and (24) demonstrate that ε_E is the elastic buckling strain: suppose the elastic material has a Young modulus of E, then

$$\sigma_1 = E\varepsilon_1$$

$$\sigma_2 = E\varepsilon_2$$

and

$$\sigma_T = E\varepsilon_T - \frac{\varepsilon_T}{\varepsilon_E}\sigma_L.$$

If $\sigma_T = 0$, then either $\varepsilon_T = 0$ or $\sigma_L = E\varepsilon_E$. Just before buckling, when $\varepsilon_T = 0$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon_L = \varepsilon_E$, the elastic buckling load of the member, P_E, is given by

$$P_E = EA\varepsilon_E = \frac{2\pi EAb^2}{L^2} = \frac{2\pi EI}{L^2}$$

where I is the second moment of area, or inertia, of the midspan section. This compares with Euler's elastic buckling load for a non-tapering member of $\pi^2 EI/L^2$: the taper reduces the buckling load by about 36%.

The material's response may be expressed as

$$\varepsilon_M = \varepsilon_M(\sigma_M). \tag{25}$$

Following Calladine's method of introducing out-of-straightness to Shanley's column [2], an initial out-of-straightness of $\varepsilon_T = \varepsilon_0$ is introduced by shifting the stress/strain curve of each flange so that

$$\varepsilon_1 = \varepsilon_M(\sigma_1) + \varepsilon_0 \tag{26}$$

$$\varepsilon_2 = \varepsilon_M(\sigma_2) - \varepsilon_0. \tag{27}$$

Provided the function ε_M gives zero strain at zero stress, when σ_1 and σ_2 are zero, ε_T equals ε_0 by Eqn (20).

The six equations (20), (21), (23), (24), (26) and (27) are particularly elegant due to the geometry of the structure and due to the substitution of stresses and strains for forces and lengths: the influence of the scale of the structure has been completely removed. The influence of proportion (i.e. stockiness) remains, due to the presence of ε_E. The remainder of the shape is defined by its out-of-straightness, ε_0, and its special lenticularity.

What of the influence of the shape of the material's stress/strain curve? Suppose Eqn (25) is rewritten

$$\frac{\varepsilon_M}{\varepsilon_C} = \phi\left(\frac{\sigma_M}{\sigma_C}\right) \tag{28}$$

where σ_C and ε_C represent critical values of stress and strain for the material (possibly maximum values) and ϕ represents what remains of the shape of the stress/strain curve. Materials with the same ϕ can have any strengths and Young moduli according to the choice of σ_C and ε_C. Normalisation of the six equations with respect to σ_C and ε_C gives

$$\frac{\varepsilon_T}{\varepsilon_C} = \frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_C} - \frac{\varepsilon_2}{\varepsilon_C}\right) \tag{29}$$

$$\frac{\varepsilon_L}{\varepsilon_C} = \frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_C} + \frac{\varepsilon_2}{\varepsilon_C}\right) + \frac{\varepsilon_C}{\varepsilon_E}\left(\frac{\varepsilon_T}{\varepsilon_C}\right)^2 \tag{30}$$

$$\frac{\sigma_L}{\sigma_C} = \frac{1}{2}\left(\frac{\sigma_1}{\sigma_C} + \frac{\sigma_2}{\sigma_C}\right) \tag{31}$$

$$\frac{\sigma_T}{\sigma_C} = \frac{1}{2}\left(\frac{\sigma_1}{\sigma_C} - \frac{\sigma_2}{\sigma_C}\right) - \frac{\varepsilon_C}{\varepsilon_E}\frac{\varepsilon_T}{\varepsilon_C}\frac{\sigma_L}{\sigma_C} \tag{32}$$

$$\frac{\varepsilon_1}{\varepsilon_C} = \phi\left(\frac{\sigma_1}{\sigma_C}\right) + \frac{\varepsilon_0}{\varepsilon_C} \tag{33}$$

$$\frac{\varepsilon_2}{\varepsilon_C} = \phi\left(\frac{\sigma_2}{\sigma_C}\right) - \frac{\varepsilon_0}{\varepsilon_C}. \tag{34}$$

It is clear from these equations that if a given set of strains and displacements can support σ_L and σ_T by σ_1 and σ_2 with a material having σ_C, then the same set of strains could support r times these loads if the material had its σ_C r times greater, supposing ϕ to remain unchanged. Therefore stresses can be scaled very simply. But this is not so for strains because of the occurrence of $\varepsilon_E/\varepsilon_C$. However, if stockiness is considered a variable which may be increased as ε_C is increased, and in the correct proportion to leave $\varepsilon_E/\varepsilon_C$ unchanged, then strains and stockiness together can be scaled just as simply as stresses.

5. PERFORMANCE OF A TWO-FLANGED ELASTOPLASTIC STRUT

The six equations contain 10 different ratios and a function ϕ expressing the shape of the material stress/strain curve. By providing one extra equation, for example

$$\frac{\sigma_T}{\sigma_C} = 0, \tag{35}$$

it is possible in general to eliminate six of the ratios and find one equation containing four of the ratios in the form

$$\frac{\varepsilon_L}{\varepsilon_C} = \psi\left(\frac{\sigma_L}{\sigma_C}, \frac{\varepsilon_E}{\varepsilon_C}, \frac{\varepsilon_0}{\varepsilon_C}\right). \tag{36}$$

For given values of $\varepsilon_E/\varepsilon_C$ and $\varepsilon_0/\varepsilon_C$ this reduces to one stress/strain curve, representing the performance of the strut. $\varepsilon_E/\varepsilon_C$ and $\varepsilon_0/\varepsilon_C$ are parameters of a family of stress/strain curves. Each parameter reflects the geometry of the strut, one its proportion and the other its out-of-straightness. This demonstrates that shape can be a "material transformer", transforming the response function ϕ of the material within the strut to a function ψ representing the response of the strut itself expressed in terms of structural stress/structural strain. Shape is an operator transforming one function into another.

Suppose ϕ is the function, shown on the left of Fig. 5, representing the shape of the stress/strain curve of an ideal elastoplastic material having strength σ_C and a Young modulus of σ_C/ε_C, and suppose this material fills a perfectly straight strut having $\varepsilon_0 = 0$. The strut will not be loaded transversely, so Eqn (35) will hold. As it is loaded axially, this strut's σ_L/ε_L response has the shape defined by ψ shown on the right of Fig. 5. There is a family of ψ-curves, parameterized by $\varepsilon_E/\varepsilon_C$, as predicted by Eqn (36). Figure 5 shows the three stages the strut may go through as it is loaded: an initial linear-elastic stage, when ψ is identical to ϕ, followed possibly by a stage of elastic buckling when ψ will branch off along a horizontal shoulder at a strain ratio of $\varepsilon_E/\varepsilon_C$ until flange 1, the more highly loaded flange, becomes plastic. Then the stress ratio in flange 1 remains equal to unity throughout the final stage as the transverse displacement continues to increase and the load in flange 2 drops. Flange 2 never becomes plastic. The middle stage of elastic buckling is missed out when $\varepsilon_E \geqslant \varepsilon_C$.

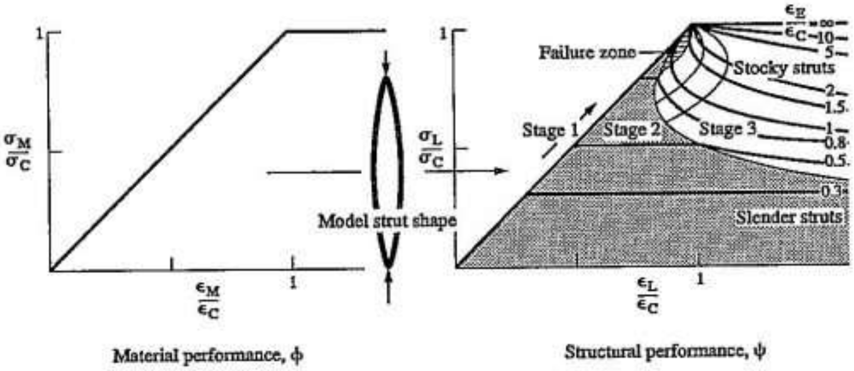

FIG. 5. The function ϕ on the left represents the shape of the stress/strain curve of an elastoplastic material. A notional compression test of a perfectly straight two-flanged model strut made of such a material would result in one of the structural performances represented by one of the family of functions ψ on the right. The particular function is determined by the relative stockiness of the strut, $\varepsilon_E/\varepsilon_C$.

During the final stage $\sigma_1/\sigma_C = 0$ and $\sigma_2/\sigma_C = \varepsilon_2/\varepsilon_C$. Using these relationships and Eqns (29), (30), (31), (32) and (35), it can be shown that during this final stage

$$\frac{\varepsilon_1}{\varepsilon_C} = \frac{2\dfrac{\varepsilon_E}{\varepsilon_C}\left(1 - \dfrac{\varepsilon_2}{\varepsilon_C}\right)}{1 + \dfrac{\varepsilon_2}{\varepsilon_C}} + \frac{\varepsilon_2}{\varepsilon_C}. \tag{37}$$

At first yield $\varepsilon_1/\varepsilon_C = 1$. Then, by Eqn (37), either $\varepsilon_2/\varepsilon_C = 1$ or $\varepsilon_2/\varepsilon_C = 2\,(\varepsilon_E/\varepsilon_C) - 1$.

For a sequence of values of $\varepsilon_2/\varepsilon_C$, $\varepsilon_1/\varepsilon_C$ may be calculated using Eqn (37). Then Eqns (29), (30) and (31) can be used to calculate σ_L/σ_C and $\varepsilon_L/\varepsilon_C$ and to plot the curves of Fig. 5.

A material stress/strain curve can never have a negative slope. A structural stress/strain curve, derived from a load/displacement curve of a member, can have negative slope, provided the member is supported by an environment which has a positive stiffness greater in magnitude than the member's own negative stiffness. When a column is free-standing and subjected to weights placed on top of it, its environment has no stiffness and the beginning of stage 3 of Fig. 5 marks the onset of static failure for any of the columns shown. At the opposite extreme, when a strut is attached at each end to what appears an infinitely stiff structure, all of stage 3 is statically accessible, except for the hatched region, where slopes of all curves are positive but falling. Within this region static failure occurs however stiff the supporting environment may be. Between these two extremes the size of the region within stage 3 corresponding to static failure depends on the stiffness of the environment. Two contours shown in Fig. 5 indicate how this region grows as the environmental stiffness decreases, first to the initial stiffness of the member itself and then to half this value, and these demonstrate that member failure is by no means tied to member strength: the failure zones do not extend to all members, and in an environment of half the member's initial stiffness, when ε_L exceeds ε_C a half-strength strut would not fail while many full-strength ones would.

The hierarchical effect of interactive buckling can be explored with this model. Walker [4] has described the interactive buckling of a stiffened plate model in which the materials remain linear elastic. He has shown that the local buckling of the main plate can be modelled by reshaping the material's stress/strain curve, flattening it as it buckles, to look like a yielding response, but not so sharply shouldered as ϕ of Fig. 5. The responses to combined buckling that he has calculated resemble the family of ψ-curves of Fig. 5, showing the same "breaking wave" shape which is characteristic of sudden instability, which occurs in his example when the geometry is "tuned" to trigger the two buckling modes together at the same load. He has commented on the sensitivity of this load in such tuned cases to the size of the initial geometric imperfections.

6. CONCLUSIONS

The lenticular pattern described in Section 3 is a basis for a family of structures whose stress and strain distributions are one-dimensional and therefore particularly easy to analyze in the difficult cases where material properties are non-linear. Complete axial load/axial displacement responses can be calculated much more easily than for uniform section members. Since member failure accompanies the vanishing of the sum of its own stiffness and its environment's stiffness, member stiffness reduction with increasing load is important information to those engineers who need to be able to predict failure loads more accurately.

The family of model structures is presented as a new analytical tool. It is hoped that its simplicity will enable us to calculate more easily how material non-linearity, geometric imperfection and interactive failure influence structural performance, and so increase our understanding of the principles of structural integrity.

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APPENDIX

Consider Eqn (1) with $y = 0$ at $x = \pm a$. Upon using the identity

$$\frac{d^2y}{dx^2} = \frac{dy}{dx}\frac{d}{dy}\left(\frac{dy}{dx}\right)$$

we find

$$yq\frac{dq}{dy} = -\frac{\pi}{2}\beta^2 \tag{A1}$$

where $q = dy/dx$. Equation (A1) can be integrated, assuming the maximum value of y is $a\beta$ and taking the constant of integration as $\pi\beta^2 \ln a\beta$, to give

$$\left(\frac{dy}{dx}\right)^2 = \pi\beta^2 \ln\frac{a\beta}{y}. \tag{A2}$$

Since the maximum occurs at $x = 0$, Eqn (A2) can be written as

$$\begin{aligned} \frac{dy}{dx} &= \left(\pi\beta^2 \ln\frac{a\beta}{y}\right)^{1/2} \quad x \leqslant 0 \\ \frac{dy}{dx} &= -\left(\pi\beta^2 \ln\frac{a\beta}{y}\right)^{1/2} \quad x \geqslant 0. \end{aligned} \tag{A3}$$

On making the substitution $u^2 = 2\ln(a\beta/y)$, these equations can again be integrated to give

$$x + a = a\sqrt{\frac{2}{\pi}}\int_u^\infty e^{-\frac{1}{2}v^2}\,dv \quad x \leqslant 0$$

$$x - a = -a\sqrt{\frac{2}{\pi}}\int_u^\infty e^{-\frac{1}{2}v^2}\,dv \quad x \geqslant 0$$

which can be combined as

$$x = \pm a[2\Phi(u) - 1] \tag{A4}$$

with $\Phi(u)$ as defined in Section 3.1.

The arc length, s, is

$$s = 2\int_{-a}^{0}\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} dx.$$

Using Eqn (A3), s can be written as

$$s = 2\int_0^{a\beta}\frac{\left(1 + \pi\beta^2\ln\frac{a\beta}{y}\right)^{1/2}}{\left(\pi\beta^2\ln\frac{a\beta}{y}\right)^{1/2}}\,dy. \tag{A5}$$

Setting $\pi\beta^2 \ln\dfrac{a\beta}{y} = w^2$ with $w = \sinh\theta$ and $2\theta = \phi$, Eqn (A5) becomes

$$s = \frac{a}{\pi\beta}e^{\frac{1}{2\pi\beta^2}}\int_0^\infty (1 + \cosh\phi)e^{-\frac{\cosh\phi}{2\pi\beta^2}}\,d\phi. \tag{A6}$$

Equation (A6) can be expressed in terms of Modified Bessel functions $K_0(z)$ and $K_1(z)$ [5]

$$s = \frac{a}{\pi\beta}e^{\frac{1}{2\pi\beta^2}}\left[K_0\left(\frac{1}{2\pi\beta^2}\right) + K_1\left(\frac{1}{2\pi\beta^2}\right)\right]. \tag{A7}$$

For large z, we have [5]

$$K_0(z) \approx \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}\left(1 - \frac{1}{8z}\right)$$
$$K_1(z) \approx \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}\left(1 + \frac{3}{8z}\right)$$

so that for small β

$$s \approx 2a\left(1 + \frac{\pi\beta^2}{4}\right).$$